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An equivalent reformulation of summability by weighted mean methods, revisited

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Abstract

We provide a correct proof of the amended theorem of our previous paper. © 2002 Elsevier Science Inc. All rights reserved.

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In [2], using matrix methods, we attempted to generalize the following result of Hardy [1].

Theorem H. *The series $\sum a_k$ of complex numbers is summable $(C, 1)$ to a finite number L if and only if the series $\sum b_n$ converges to L, where*

$$b_n := \sum_{k=n}^{\infty} \frac{a_k}{k+1}, \quad n = 0, 1, 2, \ldots$$

Theorem 1 in [2] is the following.

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Theorem MR. *Let $\overline{N}$ be the weighted mean matrix determined by a sequence $\{p_n\}$ of positive numbers such that the following conditions are satisfied*:

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \textit{and} \quad p_n/P_n \to \infty \quad \textit{as } n \to \infty, \tag{1}$$

$$\sup_{n\geqslant 0} \left\{ \frac{p_{n-1}p_{n+1}}{p_n P_{n+1}} + P_n \sum_{k=n}^{\infty} \frac{1}{P_{k+1}} \left| \frac{p_{k+1}}{p_k} - \frac{p_{k+2}P_k}{p_{k+1}P_{k+2}} \right| \right\} < \infty \tag{2}$$

and

$$\sup_{n\geqslant 0} \left\{ \frac{p_n}{p_{n+1}} + \frac{1}{P_n} \sum_{k=0}^{n} \left| \frac{p_k P_{k+1}}{p_{k+1}} - \frac{p_{k-1}P_{k-1}}{p_k} \right| \right\} < \infty \tag{3}$$

with the agreement that $p_{-1} = P_{-1} := 0$.

Then the series $\sum a_k$ is summable $\overline{N}$ to a finite number L if and only if the series $\sum b_n$ converges to L, where

$$b_n := p_n \sum_{k=n}^{\infty} \frac{a_k}{P_k}, \quad n = 0, 1, 2, \ldots \tag{4}$$

It is easy to check that the convergence of the series $\sum b_n$ to L is equivalent to the limit relation

$$\sum_{k=0}^{n} b_k = s_n + \frac{P_n}{p_{n+1}} b_{n+1} \to L \quad \text{as } n \to \infty, \quad \text{where } s_n := \sum_{k=0}^{n} a_k. \tag{5}$$

Replacing $(C, 1)$, the Cesàro matrix of order one, with the weighted mean matrix $\overline{N}$, and using the notations

$$\tau_n := \sum_{k=0}^{n} b_k, \quad \sigma_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k, \quad n = 0, 1, 2, \ldots,$$

one obtains

$$s = \Sigma a, \quad \sigma = \overline{N} s, \quad b = \overline{N}^{\mathrm{T}} a \quad \text{and} \quad \tau = \Sigma b.$$

Combining these terms formally yields $\tau = (\Sigma \overline{N}^{\mathrm{T}} \Sigma^{-1} \overline{N}^{-1})\sigma$. Since the matrix $\overline{N}^{\mathrm{T}}$ is not row finite and may not have a finite norm, associativity of multiplication need not hold, so that the proof of the statement in [2] that the summability $\overline{N}$ of $\sum a_k$ implies the convergence of $\sum b_n$ is not correct.

However, for the converse statement $\sigma = (\overline{N}\Sigma(\overline{N}^{\mathrm{T}})^{-1}\Sigma^{-1})\tau$, the matrices involved are all row finite, so associativity of multiplication holds, and the proof of the statement in [2] that the convergence of $\sum b_n$ implies the summability $\overline{N}$ of $\sum a_k$ is valid.

Now, we shall prove the following amended theorem.

Theorem 1. *Let $\overline{N}$ be the weighted mean matrix determined by a sequence $\{p_n\}$ of positive numbers satisfying the following conditions*:

(i) $p_n \geqslant a > 0$ *for* $n = 0, 1, 2, \ldots,$

(ii) $p_{n+1}/p_n = \mathrm{O}(1),$

(iii) $\{p_{n+1}/p_n P_n\}$ *is nondecreasing in* n,

(iv) $P_n \to \infty$ *as* $n \to \infty$.

If the series $\sum a_k$ is summable $\overline{N}$ to a finite number L, then the series $\sum b_n$ converges to L, where b_n is defined by (4).

Proof. Without loss of generality, we may assume that $L = 0$. Then we clearly have

$$\frac{\sigma_n'}{P_n} \to 0 \quad \text{as } n \to \infty, \quad \text{where } \sigma_n' := \sum_{k=0}^{n} p_k s_k. \tag{6}$$

Since $p_n s_n = \sigma_n' - \sigma_{n-1}'$, we also have

$$\frac{p_n s_n}{P_n} \to 0 \quad \text{as } n \to \infty. \tag{7}$$

By (4), (i) and (7), while using summation by parts, we obtain

$$\begin{aligned}
b_{n+1} &= p_{n+1} \sum_{k=n+1}^{\infty} \frac{a_k}{P_k} = p_{n+1} \lim_{m\to\infty} \sum_{k=n+1}^{m} \frac{s_k - s_{k-1}}{P_k} \\
&= p_{n+1} \lim_{m\to\infty} \left\{ \frac{s_m}{P_m} - \frac{s_n}{P_{n+1}} + \sum_{k=n+1}^{m-1} \left(\frac{1}{P_k} - \frac{1}{P_{k+1}} \right) s_k \right\} \\
&= p_{n+1} \lim_{m\to\infty} \left(\frac{1}{p_m} \frac{p_m s_m}{P_m} \right) - \frac{p_{n+1} s_n}{P_{n+1}} + p_{n+1} \sum_{k=n+1}^{\infty} c_k s_k \\
&= -\frac{p_{n+1} s_n}{P_{n+1}} + p_{n+1} \sum_{k=n+1}^{\infty} c_k s_k, \quad \text{where } c_k := \frac{1}{P_k} - \frac{1}{P_{k+1}}.
\end{aligned} \tag{8}$$

It follows from (5), (7) and (8) that

$$\begin{aligned}
\sum_{k=0}^{n} b_k &= s_n + \frac{P_n}{p_{n+1}} b_{n+1} = \left(1 - \frac{P_n}{P_{n+1}} \right) s_n + P_n \sum_{k=n+1}^{\infty} c_k s_k \\
&= \frac{s_n p_{n+1}}{P_{n+1}} + P_n \sum_{k=n+1}^{\infty} c_k s_k.
\end{aligned}$$

By (7) and (ii), we see that

$$\frac{s_n p_{n+1}}{P_{n+1}} = \frac{s_n p_n}{P_n} \frac{p_{n+1}}{p_n} \frac{P_n}{P_{n+1}} \to 0 \quad \text{as } n \to \infty.$$

To sum up, we conclude that

$$\sum_{k=0}^{n} b_k \to 0 \quad \text{if and only if} \quad P_n \sum_{k=n+1}^{\infty} c_k s_k \to 0 \quad \text{as } n \to \infty. \tag{9}$$

Applying a summation by parts again gives

$$\begin{aligned} P_n \sum_{k=n+1}^{\infty} c_k s_k &= P_n \lim_{m\to\infty} \sum_{k=n+1}^{m} c_k \frac{\sigma'_k - \sigma'_{k-1}}{p_k} \\ &= P_n \lim_{m\to\infty} \left\{ \frac{\sigma'_m c_m}{p_m} - \frac{\sigma'_n c_{n+1}}{p_{n+1}} + \sum_{k=n+1}^{m-1} \left(\frac{c_k}{p_k} - \frac{c_{k+1}}{p_{k+1}} \right) \sigma'_k \right\} \\ &= -\frac{P_n \sigma'_n p_{n+2}}{p_{n+1} P_{n+1} P_{n+2}} + P_n \sum_{k=n+1}^{\infty} P_k \left(\frac{c_k}{p_k} - \frac{c_{k+1}}{p_{k+1}} \right) \frac{\sigma'_k}{P_k}, \end{aligned} \tag{10}$$

since by (6) and (i) we have

$$\lim_{m\to\infty} \frac{\sigma'_m c_m}{p_m} = \lim_{m\to\infty} \left(\frac{\sigma'_m}{P_m} \frac{p_{m+1}}{p_m P_{m+1}} \right) = 0.$$

We define an upper triangular matrix $D = [d_{nk}]$ whose nonzero entries are given by

$$d_{nk} := \begin{cases} -\frac{p_{n+2} P_n^2}{p_{n+1} P_{n+1} P_{n+2}} & \text{for } k = n, \\ P_n P_k \left(\frac{c_k}{p_k} - \frac{c_{k+1}}{p_{k+1}} \right) & \text{for } k > n. \end{cases} \tag{11}$$

It follows from (iii) that $d_{nk} > 0$ for $k > n$ (see the definition of c_k at the end of (8)). By (ii) and (iv), we have

$$\begin{aligned} \sum_{k=n}^{\infty} |d_{nk}| &= \frac{p_{n+2} P_n^2}{p_{n+1} P_{n+1} P_{n+2}} \\ &\quad + P_n \lim_{m\to\infty} \left\{ \sum_{k=n+1}^{m} \left(\frac{P_k c_k}{p_k} - \frac{P_{k+1} c_{k+1}}{p_{k+1}} \right) + \sum_{k=n+1}^{m} c_{k+1} \right\} \\ &= \mathrm{O}(1) + P_n \lim_{m\to\infty} \left\{ \frac{P_{n+1} c_{n+1}}{p_{n+1}} - \frac{P_{m+1} c_{m+1}}{p_{m+1}} + \frac{1}{P_{n+1}} - \frac{1}{P_{m+1}} \right\} \\ &= \mathrm{O}(1) + P_n \left\{ \frac{P_{n+1} c_{n+1}}{p_{n+1}} + \frac{1}{P_{n+1}} \right\} \\ &= \mathrm{O}(1) + \frac{p_{n+2} P_n}{p_{n+1} P_{n+2}} + \frac{P_n}{P_{n+1}} = \mathrm{O}(1), \quad n = 0, 1, 2, \ldots \end{aligned} \tag{12}$$

That is, the matrix D has a finite norm. Since D is upper triangular, it clearly has zero column limits. Thus, D transforms zero sequences into zero sequences. Combining (6), (9) and (10) with what has been said just above yields the convergence of the series $\sum b_k$ to $L = 0$. This completes the proof of Theorem 1. $\square$

Actually, we also have

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} d_{nk} = 1,$$

so D is a regular summability matrix. In fact, taking into account definition (11), the calculations in (12) (see especially the last line there) and condition (iv), we find that

$$\begin{aligned}\sum_{k=n}^{\infty} d_{nk} &= -\frac{p_{n+2}P_n^2}{p_{n+1}P_{n+1}P_{n+2}} + \frac{p_{n+2}P_n}{p_{n+1}P_{n+2}} + \frac{P_n}{P_{n+1}} \\ &= \frac{P_n}{P_{n+1}}\left(\frac{p_{n+2}}{P_{n+2}} + 1\right) \to 1 \quad \text{as } n \to \infty.\end{aligned}$$

We also note that one can use a summability argument to prove the validity of the converse of Theorem 1. Furthermore, we mention the following:

Corollary 2. *Let $\overline{N}$ be the weighted mean matrix determined by the sequence*

$$p_n := (n+1)^\alpha \quad or \quad p_n := [\log(n+2)]^\alpha, \quad n = 0, 1, 2, \ldots,$$

where $\alpha \geqslant 0$. Then the series $\sum a_k$ is summable $\overline{N}$ to a finite number L if and only if the series $\sum b_n$ converges to L, where b_n is defined by (4).

In the particular case where $\alpha = 0$, Corollary 2 coincides with Theorem H proved by Hardy [1].

U. Stadtmüller (University of Ulm, Germany) pointed out that the necessity part of Theorem 1 in [2] was incorrect. His counterexample was simplified by the first author as follows. Define the sequence $\{a_k\}$ by

$$a_k := (-1)^k(2k+1), \quad k = 0, 1, 2, \ldots$$

Clearly, we have

$$s_n := \sum_{k=0}^{n} a_k = (-1)^n(n+1), \quad n = 0, 1, 2, \ldots$$

Let $p_n := 1/(n+1)$. Then the weighted mean method $\overline{N}$ defined by $\{p_n\}$ is the well-known harmonic summability method and $P_n/\ln n \to 1$ as $n \to \infty$. It is easy to verify that the sequence $\{p_n\}$ satisfies the conditions of Theorem MR, that is, conditions (1)–(3).

Since $p_k s_k = (-1)^k$ for $k = 0, 1, 2, \ldots,$ the series $\sum a_k$ is summable $\overline{N}$ to 0. On the other hand, b_n is not defined since this time the series in (4) is divergent. In fact, the general term a_k/P_k of the series in (4) does not converge to 0 as $k \to \infty$.

We note that condition (i) of Theorem 1 of this paper rules out the above example, which demonstrates that condition (i) cannot be removed from the hypotheses of Theorem 1. Furthermore, we observe that conditions (ii) and (iii) of Theorem 1 imply conditions (2) and (3) of Theorem MR.

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